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# Energy efficiency of consecutive fragmentation processes

Joaquín Fontbona, Nathalie Krell, Servet Martínez

February 18, 2010

## Abstract

We present a first study on the energy required to reduce a unit mass fragment by consecutively using several devices, as it happens in the mining industry. Two devices are considered, which we represent as different stochastic fragmentation processes. Following the self-similar energy model introduced by Bertoin and Martínez [7], we compute the average energy required to attain a size  $\eta_0$  with this two-device procedure. We then asymptotically compare, as  $\eta_0$  goes to 0 or 1, its energy requirement with that of individual fragmentation processes. In particular, we show that for certain range of parameters of the fragmentation processes and of their energy cost-functions, the consecutive use of two devices can be asymptotically more efficient than using each of them separately, or conversely.

Keywords: fragmentation process, fragmentation energy, subordinators, Laplace exponents.

Mathematics Subject Classification (2000): Primary 60J85, Secondary 60J80.

## 1 Introduction

The present work is motivated by the mining industry, where mechanical devices are used to break rocks in order to liberate the metal contained in them. This fragmentation procedure

is carried out in a series of steps (the first of them being blasting, followed then by crushers, grinders or mills) until fragments attain a sufficiently small size for the mining purposes. One of the problems that faces the mining industry is to minimize the total amount of energy consumed in this process. To be more precise, at each intermediate step, material is broken by a repetitive mechanism until particles can go across a classifying-grid leading to the next step. The output sizes are known to be not optimal in terms of the global energy cost. Moreover, since crushers or mills are large and hardly replaceable machines, those output sizes are in practice one of the few parameters on which a decision can be made.

In an idealized setting, the problem might be posed as follows: suppose that a unit-size fragment is to be reduced into fragments of sizes smaller than a fixed threshold  $\eta_0 \in (0, 1]$ , by passing consecutively through two different fragmentation mechanisms (for instance the first one could be constituted by the crushers and the second one by the mills). In this “two-step” fragmentation procedure, each mass fragment evolves in the first fragmentation mechanism until it first becomes smaller than  $\eta \in (\eta_0, 1]$ , at which moment it immediately enters the second mechanism. Then, the fragment continues to evolve until the first instant it becomes smaller than  $\eta_0$ , when it finally exits the system. The central question is:

(\*) what is the optimal choice for the intermediate threshold  $\eta$ ?

To formulate this problem we shall model each fragmentation mechanism by a continuous-time random fragmentation process, in which particles break independently of each other (branching property) and in a self-similar way. (For recent account and developments on the mathematical theory of fragmentation processes, we refer to Bertoin [6].) The self-similarity hypothesis agrees with observations made by the mining industry; see e.g. [8]. In particular, it is reasonable to assume that the energy required to break a block of size  $s$  into a set of smaller blocks of sizes  $(s_1, s_2, \dots)$  is of the form  $s^\beta \varphi(s_1/s, s_2/s, \dots)$ , where  $\varphi$  is a cost function and  $\beta > 0$  a fixed parameter. For example, in the so-called potential case, one has

$\varphi(s_1, s_2, \dots) = \sum_{n=1}^{\infty} s_n^{\beta} - 1$ , which corresponds to the law of Charles, Walker and Bond [8]. Within that mathematical framework, the asymptotic behavior of the energy required by a single fragmentation process in order that all fragments attain sizes smaller than  $\eta$  was studied in [7]. It was shown that the mean energy behaves as  $1/\eta^{\alpha-\beta}$  when  $\eta \rightarrow 0$ , where  $\alpha$  denotes the Malthusian exponent of the fragmentation process and where  $\alpha > \beta$  in physically reasonable cases. Therefore, the performances of two individual fragmentation processes are asymptotically comparable by means of the quantities  $\alpha - \beta$  and  $\hat{\alpha} - \hat{\beta}$ , where  $\hat{\alpha} > \hat{\beta}$  are the parameters associated with a second fragmentation process.

We shall formulate problem (\*) in mathematical terms adopting the same mean energy point of view as in [7]. First, we will explicitly compute the objective function, which we express in terms of the Levy and renewal measures associated with the “tagged fragment” of each of the two fragmentation processes (see [5]). Then, our goal will be to study a preliminary question related to (\*), which is weaker but still relevant for the mining industry:

(\*\*) when is the above described “two-step” procedure efficient in terms of mean energy, compared to the “one-step” procedures where only the first or only the second fragmentation mechanisms reduce a unit size fragment to fragments not larger than  $\eta_0$ ?

We shall address this question in asymptotic regimes, namely for  $\eta$  and  $\eta_0$  going together either to 0 or to 1. In both cases, we will give explicit estimates in terms of  $\eta$  for the efficiency gain or loss of using the two-step procedure.

As we shall see, if  $\alpha, \beta, \hat{\alpha}$  and  $\hat{\beta}$  are different, for any values of  $\eta/\eta_0 \in (0, 1)$  the relations between those four parameters determine the relative efficiency between the first, the second, and the two-step fragmentation procedures if  $\eta$  is sufficiently small. In particular, when  $\alpha > \hat{\alpha}$  and  $\beta > \hat{\beta}$  the answer to question (\*\*) is affirmative for  $\eta$  sufficiently small, so that the solution to problem (\*) is in general non trivial.

We shall carry out a similar analysis for large (that is, close to unit-size) thresholds. In order

to quantify the comparative efficiency of the two-step procedure, we shall make an additional hypothesis of regular variation at  $\infty$  of the Levy exponents of the tagged fragment processes. This will be transparently interpreted in terms of the infinitesimal average energy required by each of the fragmentation processes to break arbitrarily close to unit-size fragments. We will show that at least for small values of  $\log \eta_0 / \log \eta$  and variation indexes in  $(0, \frac{1}{2}]$  for both fragmentation processes, the relative infinitesimal efficiency of the two fragmentation processes determines the comparative efficiency of the three alternative fragmentation procedures if  $\eta$  is sufficiently close to 1.

We point out that the relevant parameters involved in our analysis could in principle be statistically estimated. A first concrete step in that direction has been made by Hoffmann and Krell [9] who asymptotically estimate the Levy measure of the tagged fragment from the observations of the sizes fragments at the first time they become smaller than  $\eta \rightarrow 0$ . Although this is in general not enough to recover the characteristics of the fragmentation process, it provides all the relevant parameters we need which are not observable by other means.

The remainder of this paper is organized as follows. In Section 2 we recall the construction of homogeneous fragmentation processes in terms of Poisson point processes, we describe our model of the two-step fragmentation procedure and compute its average energy using first passage laws for subordinators. In Section 3 we recall some results on renewal theory for subordinators and use them to study the small thresholds asymptotics of our problem in Theorems 1 and 2, where the two-step procedure is respectively compared with the first and the second fragmentation processes. The comparative efficiency of the three alternatives according to the values of  $\alpha, \beta, \hat{\alpha}$  and  $\hat{\beta}$  is summarized in Corollary 3. In Section 4 we introduce the idea of relative “infinitesimal efficiency” of two fragmentation procedures. We relate it to a regular variation assumption at infinity for the Levy exponent of tagged fragment, and use

it to analyze the comparative efficiency of the two-step fragmentation procedure for close to unit-size fragments, using Dynkin-Lamperti asymptotics for subordinators at first passage.

## 2 The model

### 2.1 The fragmentation process

We shall model the fragmentation mechanisms as a homogeneous fragmentation processes, as introduced in [6]. This is a homogeneous Markov process  $\mathbf{X} = (X(t, \mathbf{x}) : t \geq 0)$  taking values in

$$\mathcal{S}^\downarrow := \left\{ \mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

which satisfies the two fundamental properties of homogeneity and branching. The parameter  $\mathbf{x} = (x_1, x_2, \dots)$  is an element of  $\mathcal{S}^\downarrow$  standing for the initial condition:  $X(0, \mathbf{x}) = \mathbf{x}$  a.s.. In the case  $\mathbf{x} = (1, 0, \dots)$  we simply write  $X(t) = X(t, \mathbf{x})$ ,  $t \geq 0$ .

We observe that homogeneous fragmentation processes are self-similar fragmentation processes with zero index of self-similarity (see [6]). Since self-similar fragmentation processes with different indexes are related by a family of random time-changes (depending on fragments), there is no loss of generality in working here in the homogeneous case as the quantities we study are only size-dependent (see also [7]).

We assume that no creation of mass occurs. It is known that in this case, the process  $\mathbf{X}$  is entirely characterized by an erosion coefficient  $c \geq 0$  and a dislocation measure  $\nu$ , which is a measure on  $\mathcal{S}^\downarrow$  satisfying the conditions

$$\nu(\{1, 0, 0, \dots\}) = 0 \text{ and } \int_{\mathcal{S}^\downarrow} (1 - s_1) \nu(d\mathbf{s}) < \infty. \quad (1)$$

Moreover, we suppose that we are in the dissipative case  $\sum_{i=1}^{\infty} s_i \leq 1$  a.s., and we assume

absence of erosion:  $c = 0$ .

Let us recall the construction of a homogeneous fragmentation process in this setting, in terms the atoms of a Poisson point process (see [1]). Let  $\nu$  be a dislocation measure fulfilling conditions (1). Let  $\mathbf{K} = ((\Delta(t), k(t)) : t \geq 0)$  be a Poisson point process with values in  $\mathcal{S}^\downarrow \times \mathbb{N}$ , and with intensity measure  $\nu \otimes \sharp$ , where  $\sharp$  is the counting measure on  $\mathbb{N}$ . As in [1], we can construct a unique  $\mathcal{S}^\downarrow$ -valued process  $\mathbf{X} = (X(t, \mathbf{x}) : t \geq 0)$  started from  $\mathbf{x}$  with paths that jump only at instants  $t \geq 0$  at which a point  $(\Delta(t) = (\Delta_1, \Delta_2, \dots), k(t))$  occurs. Plainly,  $X(t, \mathbf{x})$  is obtained by replacing the  $k(t)$ -term  $X(t-, \mathbf{x})$  by the decreasing rearrangement of the sequence  $X_1(t-, \mathbf{x}), \dots, X_{k-1}(t-, \mathbf{x}), X_k(t-, \mathbf{x})\Delta_1, X_k(t-, \mathbf{x})\Delta_2, \dots, X_{k+1}(t-, \mathbf{x}), \dots$ .

Define

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}^\downarrow} \sum_{j=2}^{\infty} s_j^p \nu(ds) < \infty \right\}$$

and for every  $q \in (\underline{p}, \infty)$  consider,

$$\kappa(q) := \int_{\mathcal{S}^\downarrow} \left( 1 - \sum_{j=1}^{\infty} s_j^q \right) \nu(ds). \quad (2)$$

In the sequel, we assume the Malthusian hypothesis:  $\exists! \alpha \geq \underline{p}$  such that  $\kappa(\alpha) = 0$  which is called the Malthusian exponent.

A key tool in fragmentation theory is the tagged fragment associated with  $\mathbf{X}$ . For the precise definition, we refer the reader to [5]. The tagged fragment is a process defined by

$$\chi(t) := X_{J(t)}(t)$$

where  $J(t)$  is a random integer such that, conditioned on  $X(t)$ ,  $\mathbb{P}(J(t) = i | X(t)) = X_i(t)$  for all  $i \geq 1$ , and  $\mathbb{P}(J(t) = 0 | X(t)) = 1 - \sum_{i=1}^{\infty} X_i(t)$ .

Is is shown by Bertoin (Theorem 3 in [5]) that the process

$$\xi_t = -\log \chi(t)$$

is a subordinator. Moreover, its Laplace exponent  $\phi$  is given by

$$\phi(q) := \kappa(q + 1)$$

for  $q > \underline{p} - 1$ . Since  $\phi(\alpha - 1) = 0$ , the process  $e^{(1-\alpha)\xi(t)}$  is a nonnegative martingale, and we can then define a probability measure  $\tilde{\mathbb{P}}$  on the path space by

$$d\tilde{\mathbb{P}}|_{\mathcal{F}_t} = e^{(1-\alpha)\xi(t)} d\mathbb{P}|_{\mathcal{F}_t}, \quad (3)$$

where  $(\mathcal{F}_t : t \geq 0)$  denotes the natural filtration of  $\xi$ . It is well known that under this “tilted” law,  $\xi$  is a subordinator with Laplace exponent

$$\tilde{\phi}(q) = \phi(q + \alpha - 1). \quad (4)$$

We will respectively denote by  $\Pi$  and  $U$  the Lévy measure and the renewal measure of  $\xi_t$  under  $\tilde{\mathbb{P}}$  (see e.g. [2]).

For  $\eta \in (0, 1]$  we denote by

$$T_\eta := \inf\{t \geq 0 : \xi_t > \log(1/\eta)\}$$

the first time that the size of the tagged fragment is smaller than  $\eta$ .



## 2.2 The fragmentation energy

Following [7], we shall assume that the energy needed to split a fragment of size  $x \in [0, 1]$  into a sequence  $x_1 \geq x_2 \geq \dots$  is given by the formula

$$x^\beta \varphi \left( \frac{x_1}{x}, \frac{x_2}{x}, \dots \right),$$

where  $\beta > 0$  is a fixed constant and  $\varphi : \mathcal{S} \rightarrow \mathbb{R}$  is a measurable “cost function” such that  $\varphi((1, 0, \dots)) = 0$ .

We are interested in the total energy  $E^{(\mathbf{x})}(\eta)$  used in splitting the initial fragment of size  $x$  until each of them has reached, for the first time, a size that is smaller than  $\eta$ . This quantity is given by

$$E^{(\mathbf{x})}(\eta) = \sum_{t \geq 0} \mathbb{1}_{X_{k(t)}(t_-, \mathbf{x}) \geq \eta} X_{k(t)}^\beta(t_-, \mathbf{x}) \varphi(\Delta(t)).$$

We shall simply write

$$E(\eta) := E^{(1, 0, \dots)}(\eta).$$

The following consequence of the homogeneity property will be useful.

**Lemma 1** *Let  $\mathbf{x} = (x_1, x_2, \dots) \in \mathcal{S}^\downarrow$  and  $\eta \in [0, 1]$ . We have*

$$E^{(\mathbf{x})}(\eta) \stackrel{(law)}{=} \sum_i \mathbb{1}_{x_i \geq \eta} x_i^\beta E_i(\eta/x_i), \quad (5)$$

where for each  $i \geq 1$ ,  $E_i(\cdot)$  is the energy of a fragmentation process  $\mathbf{X}^{(i)}$  issued from  $(1, 0, \dots)$  with the same characteristics as  $\mathbf{X}$ , and the copies  $(\mathbf{X}^{(i)} : i \geq 1)$  are independent.

**Proof.** Let  $((\Delta_i(t), k_i(t)) : t \geq 0)$ ,  $i \geq 1$ , be i.i.d. Poisson point processes with intensity measure  $\nu \otimes \sharp$ . Denote by  $\overline{\mathbf{X}}^{(x_i)}$ ,  $i \geq 1$ , the sequence of independent homogeneous fragmentation processes constructed from the latter processes, respectively starting from  $(x_i, 0, \dots)$ .

From the branching property of  $\mathbf{X}$ , we have the identity

$$\mathbb{E}^{(\mathbf{x})}(\eta) \stackrel{(law)}{=} \sum_i \sum_{t \geq 0} \mathbb{1}_{x_i \geq \eta} \mathbb{1}_{\overline{X}_{k_i(t)}^{(x_i)}(t_-) \geq \eta} (\overline{X}_{k_i(t)}^{(x_i)})^\beta(t_-) \varphi(\Delta_i(t)).$$

Denoting now by  $((\Delta^{(i)}(t), k^{(i)}(t)) : t \geq 0)$ ,  $i \geq 1$ , the family of i.i.d. Poisson point processes associated with the process  $\mathbf{X}^{(i)}$ , we get by homogeneity that

$$\mathbb{E}^{(\mathbf{x})}(\eta) \stackrel{(law)}{=} \sum_i \sum_{t \geq 0} \mathbb{1}_{x_i \geq \eta} \mathbb{1}_{x_i X_{k^{(i)}(t)}^{(i)}(t_-) \geq \eta} x_i^\beta (X_{k^{(i)}(t)}^{(i)})^\beta(t_-) \varphi(\Delta^{(i)}(t)),$$

and the statement follows. ■

### 2.3 The energy of a two-step fragmentation procedure

To formulate our problem, we introduce a second Poisson point process  $\widehat{\mathbf{K}} = ((\widehat{\Delta}(t), \widehat{k}(t)), t \geq 0)$  with values in  $\mathcal{S}^\downarrow \times \mathbb{N}$ , and with intensity measure  $\widehat{\nu} \otimes \sharp$ , where  $\widehat{\nu}$  is a dislocation measure satisfying the same type of assumptions as  $\nu$ . We can then simultaneously define a family of fragmentation processes  $\widehat{\mathbf{X}} = (\widehat{X}(t, \mathbf{x}) : t \geq 0)$  indexed by the initial condition  $\mathbf{x} = (x_1, x_2, \dots)$ . We denote by  $\widehat{\alpha}$  the Malthus coefficient of  $\widehat{\nu}$ . The energy used in the second fragmentation process is assumed to take the same form as for the first, in terms of (possibly different) parameters  $\widehat{\beta}$  and  $\widehat{\varphi}$ .

We assume that  $\mathbf{K}$  and  $\widehat{\mathbf{K}}$  are independent, so the families of fragmentation processes  $\mathbf{X}$  and  $\widehat{\mathbf{X}}$  are independent, and they are called respectively the first and the second fragmentation processes.

In the sequel we assume that the first fragmentation process  $\mathbf{X}$  is issued from the unitary fragment  $(1, 0, \dots)$ . Let  $1 \geq \eta \geq \eta_0 > 0$ . We let each mass fragment evolve in the first

fragmentation process until the instant it first becomes smaller than  $\eta$ . Then it immediately enters the second fragmentation process  $\widehat{\mathbf{X}}$ , and then evolves until it first becomes smaller than  $\eta_0$ .

For each  $\eta \in (0, 1]$  let  $\mathbf{x}^\eta \in \mathcal{S}^\downarrow$  be the mass partition given by the “output” of  $\mathbf{X}$  when each of the fragments reaches for the first time a size smaller than  $\eta$ . More precisely, each fragment is “frozen” at that time, while other (larger than  $\eta$ ) fragments continue their independent evolutions. We write

$$\mathbf{x}^\eta = (x_1^\eta, x_2^\eta, \dots) \quad (6)$$

for the decreasing rearrangement of the (random) frozen sizes of fragments when exiting the first fragmentation process. By the homogeneity and branching properties, if  $\mathcal{E}(\eta, \eta_0)$  denotes the total energy spent in reducing the unit-size fragment by these procedure, we have the identity

$$\mathcal{E}(\eta, \eta_0) \stackrel{(law)}{=} \mathbf{E}(\eta) + \widehat{\mathbf{E}}^{(\mathbf{x}^\eta)}(\eta_0), \quad (7)$$

where  $\widehat{\mathbf{E}}^{(\mathbf{x})}(\cdot)$  is the energy of a copy of the second fragmentation process  $\widehat{\mathbf{X}}$  starting from  $\mathbf{x}$ , independent of the first fragmentation process.

**Remark 1** *Notice that  $\mathcal{E}(1, \eta_0)$  is the energy required to initially dislocate the unit mass with the first fragmentation process, and then use the second fragmentation process to continue breaking its fragments if their sizes are larger or equal to  $\eta_0$  (the other ones immediately exit from the system). We will denote  $\mathcal{E}(1^+, \eta_0) = \widehat{\mathbf{E}}(\eta_0)$  the total energy required when only the second fragmentation process is used from the beginning.*

*For the quantity  $\mathcal{E}(\eta_0, \eta_0) = \mathbf{E}(\eta_0)$  no confusion arises: it corresponds to the case when the first fragmentation process is used during the whole procedure.*

Our goal now is to compute the expectation of  $\mathcal{E}(\eta, \eta_0)$ .

The notation  $\widehat{\xi}$ ,  $\widehat{T}_\eta$ ,  $\widehat{\Pi}$ ,  $\widehat{U}$  and so on, will be used for the analogous objects associated with the fragmentation process  $\widehat{\mathbf{X}}$ .

So far the notation  $\mathbb{P}$  has been used to denote the law of  $\xi$ . In all the sequel, we keep the same notation  $\mathbb{P}$  to denote the product law of independent copies of the processes  $\xi$  and  $\widehat{\xi}$  in the product path space. Extending accordingly the definition in (3), we will also denote by  $\widetilde{\mathbb{P}}$  the product measure the first marginal of which is given by  $d\widetilde{\mathbb{P}}|_{\mathcal{F}_t} = e^{(1-\alpha)\xi(t)}d\mathbb{P}|_{\mathcal{F}_t}$  and the second one given by  $d\widetilde{\mathbb{P}}|_{\widehat{\mathcal{F}}_t} = e^{(1-\widehat{\alpha})\widehat{\xi}(t)}d\mathbb{P}|_{\widehat{\mathcal{F}}_t}$ . Here  $(\mathcal{F}_t : t \geq 0)$  and  $(\widehat{\mathcal{F}}_t : t \geq 0)$  are the natural filtrations of  $\xi$  and  $\widehat{\xi}$  respectively.

We shall assume throughout that the following integrability condition holds:

$$\varphi \in L^1(\nu) \quad \text{and} \quad \widehat{\varphi} \in L^1(\widehat{\nu}). \quad (8)$$

In this case we define

$$C = \int_S \varphi(\mathbf{s})\nu(d\mathbf{s}) \quad \text{and} \quad \widehat{C} = \int_S \widehat{\varphi}(\mathbf{s})\widehat{\nu}(d\mathbf{s}).$$

Let us introduce the functions

$$\Psi(x) = C \int_0^x e^{(\alpha-\beta)y} U(dy), \quad \widehat{\Psi}(x) = \widehat{C} \int_0^x e^{(\widehat{\alpha}-\widehat{\beta})y} \widehat{U}(dy), \quad x \geq 0.$$

To simplify the notation we will put

$$\forall a > 0 : \ell(a) := \log(1/a).$$

We have the elements to compute the expected energy requirement in the two step fragmentation procedure.

**Lemma 2** Assume that the integrability condition (8) is satisfied. Let  $\eta_0 \in (0, 1)$ . Then, we have for  $\eta_0 < \eta < 1$  that

$$\begin{aligned}\mathbb{E}(\mathcal{E}(\eta, \eta_0)) &= C \int_0^{\ell(\eta)} e^{(\alpha-\beta)y} U(dy) \\ &\quad + \widehat{C} \int_0^{\ell(\eta)} \int_{\ell(\eta)-y}^{\ell(\eta_0)-y} e^{(\alpha-\widehat{\beta})(z+y)} \left[ \int_0^{\ell(\eta_0)-(z+y)} e^{(\widehat{\alpha}-\widehat{\beta})x} \widehat{U}(dx) \right] \Pi(dz) U(dy) \\ &= \Psi(\ell(\eta)) + \widetilde{\mathbb{E}} \left( \mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha-\widehat{\beta})\xi_{T_\eta}} \widehat{\Psi}(\ell(\eta_0) - \xi_{T_\eta}) \right),\end{aligned}$$

and

$$\mathbb{E}(\mathcal{E}(\eta_0, \eta_0)) = \mathbb{E}(\mathcal{E}(\eta_0)) = \Psi(\ell(\eta_0)), \quad \mathbb{E}(\mathcal{E}(1^+, \eta_0)) = \mathbb{E}(\widehat{\mathcal{E}}(\eta_0)) = \widehat{\Psi}(\ell(\eta_0)).$$

When the renewal measures  $U(dx)$  has no atom at 0 one has

$$\mathbb{E}(\mathcal{E}(1^+, \eta_0)) = \mathbb{E}(\mathcal{E}(1, \eta_0)).$$

**Proof.** The proof is an extension of arguments given in [7] corresponding to the case “ $\eta = 1^+$ ” or  $\eta = \eta_0$  and which we repeat here for convenience. By the compensation formula for the Poisson point process  $(\Delta(u), k(u))$  associated with the first fragmentation process  $\mathbf{X}$ , we get that for  $\eta_0 \in (0, 1]$ ,

$$\mathbb{E}(\mathcal{E}(\eta_0)) = \mathbb{E} \left( \int_0^\infty \mathbb{1}_{\chi(t) > \eta_0} (\chi(t))^{\beta-1} dt \right) \int_S \varphi(\mathbf{s}) \nu(d\mathbf{s}) = C \mathbb{E} \left( \int_0^\infty \mathbb{1}_{\xi_t < \ell(\eta_0)} e^{(1-\beta)\xi_t} dt \right).$$

Thus

$$\mathbb{E}(\mathcal{E}(\eta_0)) = C \widetilde{\mathbb{E}} \left( \int_0^\infty \mathbb{1}_{\xi_t < \ell(\eta_0)} e^{(\alpha-\beta)\xi_t} dt \right) = C \int_0^{\ell(\eta_0)} e^{(\alpha-\beta)y} U(dy) = \Psi(\ell(\eta_0)). \quad (9)$$

Similarly,

$$\mathbb{E}(\widehat{\mathcal{E}}(\eta_0)) = \widehat{C} \int_0^{\ell(\eta_0)} e^{(\widehat{\alpha}-\widehat{\beta})y} \widehat{U}(dy) = \widehat{\Psi}(\ell(\eta_0)).$$

The above identity also implies that  $\mathbb{E}(\mathcal{E}(1, \eta_0)) = \widehat{C} \int_{0^+}^{\ell(\eta_0)} e^{(\widehat{\alpha}-\widehat{\beta})y} \widehat{U}(dy) = \mathbb{E}(\widehat{\mathbf{E}}(\eta_0))$  when  $U$  has no atom at 0.

The statement is thus proved for the cases “ $\eta = 1^+$ ” and  $\eta = \eta_0$ . For the general case, we use Lemma 1 to get

$$\begin{aligned} \mathbb{E}(\widehat{\mathbf{E}}^{(\mathbf{x}_\eta)}(\eta_0)) &= \mathbb{E} \left( \sum_i \mathbb{1}_{x_{\eta,i} > \eta_0} x_{\eta,i}^{\widehat{\beta}} \widehat{\mathbf{E}}_i(\eta_0/x_{\eta,i}) \right) \\ &= \mathbb{E} \left( \sum_i \mathbb{1}_{x_{\eta,i} > \eta_0} x_{\eta,i}^{\widehat{\beta}} \mathbb{E}(\widehat{\mathbf{E}}_i(\eta_0/x_{\eta,i}) \mid x_{\eta,i}) \right) \\ &= \mathbb{E} \left( \mathbb{1}_{\chi(T_\eta) > \eta_0} (\chi(T_\eta))^{\widehat{\beta}-1} \widehat{\mathbf{E}}(\widehat{\mathbf{E}}(\eta_0/y))|_{y=\chi(T_\eta)} \right), \end{aligned}$$

where  $\widehat{\mathbf{E}}(\cdot)$  is the energy of a copy of the second fragmentation process, starting from the unit mass, and which is independent of the first one, and  $\widehat{\mathbf{E}}_i(\cdot)$  are independent copies of  $\widehat{\mathbf{E}}(\cdot)$ . Then, since  $\chi(t) = e^{-\xi t}$ , we have,

$$\begin{aligned} \mathbb{E}(\widehat{\mathbf{E}}^{(\mathbf{x}_\eta)}(\eta_0)) &= \widetilde{\mathbb{E}} \left( \mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha-\widehat{\beta})\xi_{T_\eta}} \widetilde{\mathbb{E}}(\widehat{\mathbf{E}}(\eta_0 e^z))|_{z=\xi_{T_\eta}} \right) \\ &= \widetilde{\mathbb{E}} \left( \mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha-\widehat{\beta})\xi_{T_\eta}} \widehat{\Psi}(\ell(\eta_0) - \xi_{T_\eta}) \right). \end{aligned}$$

According to Lemma 1.10 of [4] the distribution of  $\xi_{T_\eta}$  under  $\widetilde{\mathbb{P}}$  is given by

$$\widetilde{\mathbb{P}}(\xi_{T_\eta} \in dz) = \int_0^{\ell(\eta)} \mathbb{1}_{\ell(\eta) < z} \Pi(dz - y) U(dy).$$

Therefore,

$$\mathbb{E}(\widehat{\mathbf{E}}^{(\mathbf{x}_\eta)}(\eta_0)) = \int_0^{\ell(\eta)} \left[ \int_{\ell(\eta)-y}^{\ell(\eta_0)-y} e^{(\alpha-\widehat{\beta})(z+y)} \widehat{\Psi}(\ell(\eta) - (z+y)) \Pi(dz) \right] U(dy).$$

By bringing the pieces together and by using the identity (7) we get the result. ■

In analogy with (6), we introduce the notation

$$\widehat{\mathbf{x}}^\eta = (\widehat{x}_1^\eta, \widehat{x}_2^\eta, \dots) \quad (10)$$

for the decreasing rearrangement of the frozen sizes of fragments smaller than  $\eta$ , that exit the second fragmentation process started from the unit mass. The following decompositions of the total energy will be useful in the sequel:

**Remark 2** For  $1 \geq \eta \geq \eta_0 > 0$  we have

$$\widehat{\mathbf{E}}(\eta_0) = \widehat{\mathbf{E}}(\eta) + \widehat{\mathbf{E}}^{(\widehat{\mathbf{x}}^\eta)}(\eta_0),$$

whence,

$$\mathcal{E}(\eta, \eta_0) - \mathcal{E}(1^+, \eta_0) = \mathbf{E}(\eta) - \widehat{\mathbf{E}}(\eta) + \widehat{\mathbf{E}}^{(\mathbf{x}^\eta)}(\eta_0) - \widehat{\mathbf{E}}^{(\widehat{\mathbf{x}}^\eta)}(\eta_0).$$

From this relation and by similar computations as in Lemma 2, we can write

$$\begin{aligned} \mathbb{E}(\mathcal{E}(\eta, \eta_0) - \mathcal{E}(1^+, \eta_0)) &= \Psi(\ell(\eta)) - \widehat{\Psi}(\ell(\eta)) \\ &\quad + \widetilde{\mathbb{E}} \left( \mathbf{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha - \widehat{\beta})\xi_{T_\eta}} \widehat{\Psi}(\ell(\eta_0) - \xi_{T_\eta}) \right) \\ &\quad - \widetilde{\mathbb{E}} \left( \mathbf{1}_{\widehat{\xi}_{\widehat{T}_\eta} < \ell(\eta_0)} e^{(\widehat{\alpha} - \widehat{\beta})\widehat{\xi}_{\widehat{T}_\eta}} \widehat{\Psi}(\ell(\eta_0) - \widehat{\xi}_{\widehat{T}_\eta}) \right). \end{aligned}$$

Observe that when  $U$  has no atom at 0, one can replace  $\mathcal{E}(1^+, \eta_0)$  by  $\mathcal{E}(1, \eta_0)$  on the left hand side of the formula.

Similarly, we have

$$\begin{aligned}\mathbb{E}(\mathcal{E}(\eta, \eta_0) - \mathcal{E}(\eta_0, \eta_0)) &= \widehat{\mathbb{E}}^{(\mathbf{x}^\eta)}(\eta_0) - \mathbb{E}^{(\mathbf{x}^\eta)}(\eta_0) \\ &= \widetilde{\mathbb{E}}\left(\mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha - \widehat{\beta})\xi_{T_\eta}} \widehat{\Psi}(\ell(\eta_0) - \xi_{T_\eta}) - \mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha - \beta)\xi_{T_\eta}} \Psi(\ell(\eta_0) - \xi_{T_\eta})\right).\end{aligned}$$

### 3 Small thresholds

In this section, we consider the total energy  $\mathbb{E}(\mathcal{E}(\eta, \eta_0))$  when  $\eta_0$  and  $\eta$  go to 0 in a suitable joint asymptotics. Our goal is to compare it with the mean energy required for reducing the unit fragment to fragments smaller than  $\eta_0$  using only the first or only the second fragmentation processed. We shall assume that the quantities

$$m(\alpha) := \int_{S^\downarrow} \sum_{i=1}^{\infty} s_n^\alpha \log\left(\frac{1}{s_n^\alpha}\right) \nu(ds) \quad \text{and} \quad \widehat{m}(\widehat{\alpha}) := \int_{S^\downarrow} \sum_{i=1}^{\infty} s_n^\alpha \log\left(\frac{1}{s_n^\alpha}\right) \widehat{\nu}(ds)$$

are finite. Moreover, we impose the conditions

$$\beta < \alpha \quad \text{and} \quad \widehat{\beta} < \widehat{\alpha}.$$

The latter assumption is physically reasonable, since the energy  $\Psi(\infty)$  (respectively  $\widehat{\Psi}(\infty)$ ) required in order that all fragments vanish in the first (respectively second) fragmentation processes is otherwise finite (see Remark 1 in [7]).

The following asymptotic result on the mean energy of a single fragmentation processes is based on the renewal Theorem for subordinators (Bertoin et al. [3]). Its proof is simply adapted from that of Lemma 4 in [7], see also Theorem 1 therein.

**Lemma 3** *Under the previous assumptions, we have*

$$\lim_{\eta \rightarrow 0} \eta^{\alpha - \beta} \mathbb{E}(\mathcal{E}(\eta)) = \frac{C}{(\alpha - \beta)m(\alpha)} \quad \text{and} \quad \lim_{\eta \rightarrow 0} \eta^{\widehat{\alpha} - \widehat{\beta}} \mathbb{E}(\widehat{\mathcal{E}}(\eta)) = \frac{\widehat{C}}{(\widehat{\alpha} - \widehat{\beta})\widehat{m}(\widehat{\alpha})}.$$



By the renewal Theorem for subordinators we also have as  $\eta \rightarrow 0^+$  that

$$\tilde{\mathbb{P}}(\xi_{T_\eta} - \ell(\eta) \in du) \rightarrow M(du) := \frac{1}{m(\alpha)} \int_{\mathbb{R}^+} \Pi(y + du) dy,$$

and

$$\tilde{\mathbb{P}}(\hat{\xi}_{\hat{T}_\eta} - \ell(\eta) \in du) \rightarrow \widehat{M}(du) := \frac{1}{\widehat{m}(\widehat{\alpha})} \int_{\mathbb{R}^+} \widehat{\Pi}(y + du) dy$$

in the weak sense. Let us define, for  $\lambda > 0$  a fixed parameter, the finite and strictly positive constants

$$F_\lambda := m(\alpha) \int_0^\lambda e^{(\alpha - \widehat{\beta})u} \widehat{\Psi}(\lambda - u) M(du),$$

$$D_\lambda := m(\alpha) \int_0^\lambda e^{(\alpha - \beta)u} \Psi(\lambda - u) M(du)$$

$$\widehat{D}_\lambda := \widehat{m}(\widehat{\alpha}) \int_0^\lambda e^{(\widehat{\alpha} - \widehat{\beta})u} \widehat{\Psi}(\lambda - u) \widehat{M}(du).$$

We fix in the sequel the parameter  $\lambda > 0$ . With these elements, we are in position to explicitly study the (comparative) behavior of the total energy for small thresholds  $\eta$  and  $\eta_0$ , when these are bond by the relation

$$\eta_0 = \eta e^{-\lambda}.$$

**Theorem 1** (*Two-step procedure versus first fragmentation only*)

*Assume that the renewal measure  $U(dx)$  has no atom at 0. For any  $\lambda > 0$ , the following hold:*

(a) *If  $\widehat{\beta} > \beta$ , then  $\forall \varepsilon \in (0, \frac{\alpha - \beta}{C} D_\lambda) \exists \eta_\varepsilon \in (0, 1)$  such that*

$$\forall \eta \leq \eta_\varepsilon : \quad \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \leq \left( \varepsilon - \frac{\alpha - \beta}{C} D_\lambda \right) \mathbb{E}(\mathbb{E}(\eta)) + \mathbb{E}(\mathbb{E}(\eta e^{-\lambda})) < \mathbb{E}(\mathbb{E}(\eta e^{-\lambda})).$$

(b) If  $\widehat{\beta} < \beta$ , then  $\forall M > 0 \exists \eta_M \in (0, 1)$  such that

$$\forall \eta \leq \eta_M : \quad \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \geq M \mathbb{E}(\mathbf{E}(\eta)) + \mathbb{E}(\mathbf{E}(\eta e^{-\lambda})) > \mathbb{E}(\mathbf{E}(\eta e^{-\lambda})).$$

(c) If  $\widehat{\beta} = \beta$ , then  $\forall \varepsilon \in (0, 1) \exists \eta_\varepsilon \in (0, 1)$  such that  $\forall \eta \leq \eta_\varepsilon$ :

$$\begin{aligned} \left( -\varepsilon + \frac{\alpha - \beta}{C}(F_\lambda - D_\lambda) \right) \mathbb{E}(\mathbf{E}(\eta)) + \mathbb{E}(\mathbf{E}(\eta e^{-\lambda})) \\ \leq \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \leq \left( \varepsilon + \frac{\alpha - \beta}{C}(F_\lambda - D_\lambda) \right) \mathbb{E}(\mathbf{E}(\eta)) + \mathbb{E}(\mathbf{E}(\eta e^{-\lambda})). \end{aligned}$$

In all cases, one can replace  $\mathbb{E}(\mathbf{E}(\eta))$  by  $C [\eta^{\alpha-\beta} m(\alpha)(\alpha - \beta)]^{-1}$ .

**Proof.** All parts are obtained by taking limit when  $\eta \rightarrow 0$  in the identity

$$\begin{aligned} \eta^{\alpha-\beta} (\mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) - \mathbb{E}(\mathcal{E}(\eta e^{-\lambda}, \eta e^{-\lambda}))) = & \widetilde{\mathbb{E}} \left( e^{(\alpha-\widehat{\beta})(\xi_{T_\eta} - \ell(\eta))} \widehat{\Psi}(\lambda - (\xi_{T_\eta} - \ell(\eta)) \mathbb{1}_{\xi_{T_\eta} - \ell(\eta) < \lambda}) \right) \eta^{\widehat{\beta}-\beta} \\ & - \widetilde{\mathbb{E}} \left( e^{(\alpha-\beta)(\xi_{T_\eta} - \ell(\eta))} \Psi(\lambda - (\xi_{T_\eta} - \ell(\eta)) \mathbb{1}_{\xi_{T_\eta} - \ell(\eta) < \lambda}) \right), \end{aligned}$$

which follows from Remark 2, and then using Lemma 3 and the previously mentioned weak convergence result for  $\widetilde{\mathbb{P}} \left( \widehat{\xi}_{\widehat{T}_\eta} - \ell(\eta) \in dy \right)$  (notice that the limit is absolutely continuous).

■

**Theorem 2** (*Two-step procedure versus second fragmentation only*)

Assume that  $U(dx)$  and  $\widehat{U}(dx)$  have no atom at 0. For any  $\lambda > 0$ , the following hold:

(a) If  $\widehat{\alpha} > \alpha$ , then  $\forall \varepsilon \in (0, \frac{\widehat{\alpha}-\widehat{\beta}}{\widehat{C}} \widehat{D}_\lambda) \exists \eta_\varepsilon \in (0, 1)$  such that

$$\forall \eta \leq \eta_\varepsilon : \quad \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \leq \left( \varepsilon - \frac{\widehat{\alpha} - \widehat{\beta}}{\widehat{C}} \widehat{D}_\lambda \right) \mathbb{E}(\widehat{\mathbf{E}}(\eta)) + \mathbb{E}(\widehat{\mathbf{E}}(\eta e^{-\lambda})) < \mathbb{E}(\widehat{\mathbf{E}}(\eta e^{-\lambda})).$$

(b) If  $\hat{\alpha} < \alpha$ , then  $\forall M > 0 \exists \eta_M \in (0, 1)$  such that

$$\forall \eta \leq \eta_M : \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \geq M \mathbb{E}(\hat{\mathbf{E}}(\eta)) + \mathbb{E}(\hat{\mathbf{E}}(\eta e^{-\lambda})) > \mathbb{E}(\hat{\mathbf{E}}(\eta e^{-\lambda})).$$

(c) If  $\hat{\alpha} = \alpha$ , then  $\forall \varepsilon \in (0, 1) \exists \eta_\varepsilon$  such that  $\forall \eta \leq \eta_\varepsilon$ ,

$$\begin{aligned} \left( -\varepsilon + \frac{\hat{\alpha} - \hat{\beta}}{\hat{C}}(F_\lambda - \hat{D}_\lambda) \right) \mathbb{E}(\hat{\mathbf{E}}(\eta)) + \mathbb{E}(\hat{\mathbf{E}}(\eta e^{-\lambda})) \\ \leq \mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) \leq \left( \varepsilon + \frac{\hat{\alpha} - \hat{\beta}}{\hat{C}}(F_\lambda - \hat{D}_\lambda) \right) \mathbb{E}(\hat{\mathbf{E}}(\eta)) + \mathbb{E}(\hat{\mathbf{E}}(\eta e^{-\lambda})). \end{aligned}$$

In all cases, one can replace  $\mathbb{E}(\hat{\mathbf{E}}(\eta))$  by  $\hat{C} \left[ \eta^{\hat{\alpha} - \hat{\beta}} m(\hat{\alpha})(\hat{\alpha} - \hat{\beta}) \right]^{-1}$ .

**Proof.** The proof is similar to previous one, noting that

$$\begin{aligned} \eta^{\hat{\alpha} - \hat{\beta}} (\mathbb{E}(\mathcal{E}(\eta, \eta e^{-\lambda})) - \mathbb{E}(\mathcal{E}(1, \eta e^{-\lambda}))) &= \tilde{\mathbb{E}} \left( e^{(\hat{\alpha} - \hat{\beta})(\xi_{T_\eta} - \ell(\eta))} \hat{\Psi}(\lambda - (\xi_{T_\eta} - \ell(\eta)) \mathbb{1}_{\xi_{T_\eta} - \ell(\eta) < \lambda}) \right) \eta^{\hat{\alpha} - \alpha} \\ &\quad - \tilde{\mathbb{E}} \left( e^{(\hat{\alpha} - \hat{\beta})(\hat{\xi}_{\hat{T}_\eta} - \ell(\eta))} \Psi(\lambda - (\hat{\xi}_{\hat{T}_\eta} - \ell(\eta)) \mathbb{1}_{\hat{\xi}_{\hat{T}_\eta} - \ell(\eta) < \lambda}) \right). \end{aligned}$$

■

We next summarize the main results of this section in an asymptotic comparative scheme. The notation  $F_{1,2}$  refers to the situation where in the two-step fragmentation procedure both devices are effectively used (i.e.  $\eta_0/\eta \in (0, 1)$ ), whereas the notation  $F_1$  and  $F_2$  respectively refer to the situations where only the first or only the second fragmentation process is used.

**Corollary 3** Assume that  $U(dx)$  and  $\hat{U}(dx)$  have no atom at 0. In each of the following cases, the corresponding assertion holds true for any value of  $\eta_0/\eta \in (0, 1)$  as soon as  $\eta$  is

sufficiently small:

- $\hat{\alpha} > \alpha, \hat{\beta} < \beta$  (thus  $\alpha - \beta < \hat{\alpha} - \hat{\beta}$ ) :  $F_1$  is better than  $F_{1,2}$  which is better than  $F_2$  .
- $\hat{\alpha} < \alpha, \hat{\beta} > \beta$  (thus  $\alpha - \beta > \hat{\alpha} - \hat{\beta}$ ) :  $F_2$  is better than  $F_{1,2}$  which is better than  $F_1$  .
- $\hat{\alpha} < \alpha, \hat{\beta} < \beta$  and  $\alpha - \beta < \hat{\alpha} - \hat{\beta}$  :  $F_1$  is better than  $F_2$  which is better than  $F_{1,2}$  .
- $\hat{\alpha} < \alpha, \hat{\beta} < \beta$  and  $\alpha - \beta > \hat{\alpha} - \hat{\beta}$  :  $F_2$  is better than  $F_1$  which is better than  $F_{1,2}$  .
- $\hat{\alpha} > \alpha, \hat{\beta} > \beta$  and  $\alpha - \beta < \hat{\alpha} - \hat{\beta}$  :  $F_{1,2}$  is better than  $F_1$  which is better than  $F_2$  .
- $\hat{\alpha} > \alpha, \hat{\beta} > \beta$  and  $\alpha - \beta > \hat{\alpha} - \hat{\beta}$  :  $F_{1,2}$  is better than  $F_2$  which is better than  $F_1$  .

**Remark 3** By parts c) of Theorems 1 and 2, if  $\hat{\alpha} = \alpha$  or if  $\hat{\beta} = \beta$  the comparative efficiency of  $F_1$ ,  $F_2$  and  $F_{1,2}$  for  $\eta$  small enough is in general determined by those parameters but also by the value of  $\eta_0/\eta \in (0, 1)$ .

## 4 Close-to-unit size thresholds

We shall next be interested in the behavior of  $\mathbb{E}(\mathcal{E}(\eta, \eta_0))$  for large values of  $\eta$  and  $\eta_0$ . Again, we shall compare the mean energy of the two-step fragmentation procedure with the situations when only the second, or only the first fragmentation process is used.

We shall assume in this analysis that the subordinators  $\xi$  and  $\hat{\xi}$  satisfy under  $\tilde{\mathbb{P}}$  a condition of regular variation at  $\infty$ . Namely, respectively denoting by  $\tilde{\phi}$  and  $\hat{\phi}$  their Laplace exponents (see (4)), we assume

$$(\mathbf{RV}) \quad \exists \rho, \hat{\rho} \in (0, 1) \text{ such that } \forall \lambda \geq 0 : \quad \lim_{q \rightarrow \infty} \frac{\tilde{\phi}(\lambda q)}{\tilde{\phi}(q)} = \lambda^\rho, \quad \lim_{q \rightarrow \infty} \frac{\hat{\phi}(\lambda q)}{\hat{\phi}(q)} = \lambda^{\hat{\rho}}.$$

This assumption can be equivalently (and transparently) stated in terms of the infinitesimal

behavior near  $\eta = 1$  of the “mean energy functions”  $\eta \mapsto \mathbb{E}(\mathbf{E}(\eta))$  and  $\eta \mapsto \mathbb{E}(\widehat{\mathbf{E}}(\eta))$  of each of the fragmentation processes. See Remark 5 below.

Recall that a function  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to vary slowly at 0 if  $\lim_{x \rightarrow 0^+} G(\lambda x)/G(x) = 1$  for all  $\lambda \in [0, \infty)$ . A well known fact that will be used in the sequel is that such convergence is uniform in  $\lambda \in [0, \lambda_0]$ , for all  $\lambda_0 \in (0, \infty)$ .

By  $L$  and  $\widehat{L}$  we shall denote the nonnegative slowly varying functions at 0 defined by the relations

$$L\left(\frac{1}{x}\right) = \frac{x^\rho}{\widetilde{\phi}(x)}, \quad \widehat{L}\left(\frac{1}{x}\right) = \frac{x^{\widehat{\rho}}}{\widetilde{\widehat{\phi}}(x)}.$$

**Remark 4** *Using the aforementioned uniform convergence result for  $L$  and  $\widehat{L}$  it is not hard to check that  $\lim_{q \rightarrow \infty} \frac{\phi(q)}{\widetilde{\phi}(q)} = \lim_{q \rightarrow \infty} \frac{\widehat{\phi}(q)}{\widetilde{\widehat{\phi}}(q)} = 1$ . Consequently, (RV) implies that the same condition hold on  $\phi$  and  $\widehat{\phi}$  and conversely.*

We define

$$Q_{\phi, \widehat{\phi}} := \lim_{q \rightarrow \infty} \frac{\widehat{\phi}(q)}{\phi(q)} = \lim_{q \rightarrow \infty} \frac{\widetilde{\widehat{\phi}}(q)}{\widetilde{\phi}(q)} = \lim_{x \rightarrow 0^+} \frac{L(x)x^\rho}{\widehat{L}(x)x^{\widehat{\rho}}}$$

if the limit in  $[0, \infty]$  exists. More generally, we write

$$Q_{\phi, \widehat{\phi}}^+ := \limsup_{q \rightarrow \infty} \frac{\widehat{\phi}(q)}{\phi(q)} = \limsup_{x \rightarrow 0^+} \frac{L(x)x^\rho}{\widehat{L}(x)x^{\widehat{\rho}}}$$

and

$$Q_{\phi, \widehat{\phi}}^- := \liminf_{q \rightarrow \infty} \frac{\widehat{\phi}(q)}{\phi(q)} = \liminf_{x \rightarrow 0^+} \frac{L(x)x^\rho}{\widehat{L}(x)x^{\widehat{\rho}}}.$$

Recall the notation

$$\Psi(x) = C \int_0^x e^{(\alpha-\beta)y} U(dy), \quad \widehat{\Psi}(x) = \widehat{C} \int_0^x e^{(\widehat{\alpha}-\widehat{\beta})y} \widehat{U}(dy).$$

**Lemma 4** *We have*

$$CQ_{\phi, \hat{\phi}}^- - \hat{C} \leq \liminf_{\eta \rightarrow 1^-} \frac{\Psi(\ell(\eta)) - \hat{\Psi}(\ell(\eta))}{\hat{L}(\ell(\eta))\ell(\eta)^{\hat{\rho}}} \leq \limsup_{\eta \rightarrow 1^-} \frac{\Psi(\ell(\eta)) - \hat{\Psi}(\ell(\eta))}{\hat{L}(\ell(\eta))\ell(\eta)^{\hat{\rho}}} \leq CQ_{\phi, \hat{\phi}}^+ - \hat{C}.$$

*In particular,*

$$\lim_{\eta \rightarrow 1^-} \frac{\Psi(\ell(\eta)) - \hat{\Psi}(\ell(\eta))}{\hat{L}(\ell(\eta))\ell(\eta)^{\hat{\rho}}} = \begin{cases} \infty & \text{if } \hat{\rho} > \rho \\ -\hat{C} & \text{if } \hat{\rho} < \rho \\ CQ_{\phi, \hat{\phi}} - \hat{C} & \text{if } \hat{\rho} = \rho \text{ and } \exists Q_{\phi, \hat{\phi}} = \lim_{x \rightarrow 0^+} \frac{L(x)}{\hat{L}(x)} \in [0, \infty]. \end{cases}$$

**Proof.** By classic Tauberian theorems (see e.g. Th, 5.13 in [10] or Section 0.7 in [2]), our assumptions on  $\tilde{\phi}$  and  $\hat{\phi}$  are respectively equivalent to

$$\lim_{x \rightarrow 0^+} \frac{U(x)}{x^\rho L(x)} = 1, \quad \lim_{x \rightarrow 0^+} \frac{\hat{U}(x)}{x^{\hat{\rho}} \hat{L}(x)} = 1.$$

On the other hand, we have

$$\begin{aligned} \Psi(x) - \hat{\Psi}(x) &\leq C e^{|\alpha - \beta|x} U(x) - \hat{C} e^{-|\hat{\alpha} - \hat{\beta}|x} \hat{U}(x) \\ &= \hat{L}(x) x^{\hat{\rho}} \hat{C} \left( \frac{C}{\hat{C}} e^{|\alpha - \beta|x} \frac{U(x)}{x^\rho L(x)} \frac{L(x)}{\hat{L}(x)} x^{\rho - \hat{\rho}} - e^{-|\hat{\alpha} - \hat{\beta}|x} \frac{\hat{U}(x)}{x^{\hat{\rho}} \hat{L}(x)} \right) \end{aligned}$$

and similarly,

$$\begin{aligned} \Psi(x) - \hat{\Psi}(x) &\geq C e^{-|\alpha - \beta|x} U(x) - \hat{C} e^{|\hat{\alpha} - \hat{\beta}|x} \hat{U}(x) \\ &= \hat{L}(x) x^{\hat{\rho}} \hat{C} \left( \frac{C}{\hat{C}} e^{-|\alpha - \beta|x} \frac{U(x)}{x^\rho L(x)} \frac{L(x)}{\hat{L}(x)} x^{\rho - \hat{\rho}} - e^{|\hat{\alpha} - \hat{\beta}|x} \frac{\hat{U}(x)}{x^{\hat{\rho}} \hat{L}(x)} \right). \end{aligned}$$

The first statement follows from these bounds. To complete the proof, notice that since  $\frac{L(x)}{\hat{L}(x)}$

is slowly varying at 0, we have that

$$\lim_{x \rightarrow 0^+} \frac{L(x)}{\widehat{L}(x)} x^{\rho - \widehat{\rho}} = \begin{cases} \infty & \text{if } \widehat{\rho} > \rho \\ 0 & \text{if } \widehat{\rho} < \rho \\ Q_{\phi, \widehat{\phi}} & \text{if } \widehat{\rho} = \rho \text{ and } \exists \lim_{x \rightarrow 0^+} \frac{L(x)}{\widehat{L}(x)} \in [0, \infty], \end{cases}$$

using also the fact that  $\lim_{x \rightarrow 0^+} G(x) = 0$  for any regularly varying (at 0) function  $G(x)$  with positive index.

■

Notice that **(RV)** implies that  $U$  has no atom at 0 (see e.g. the first lines of the previous proof).

**Remark 5** *The estimates used in the proof of Lemma 4 show that*

$$\Psi(x) \sim CU(x) \quad \text{and} \quad \widehat{\Psi}(x) \sim \widehat{C}\widehat{U}(x) \quad \text{when } x \rightarrow 0^+,$$

so that  $\Psi(x) \sim Cx^\rho L(x)$  and  $\widehat{\Psi}(x) \sim \widehat{C}x^{\widehat{\rho}}\widehat{L}(x)$  as well. Consequently, by the aforementioned Tauberian results, assumption **(RV)** is equivalent to

**(RV)**  $x \mapsto \mathbb{E}(\mathbf{E}(e^{-x}))$  and  $x \mapsto \mathbb{E}(\widehat{\mathbf{E}}(e^{-x}))$  are regularly varying at  $0^+$  with indexes  $\rho, \widehat{\rho} \in (0, 1)$  respectively.

This alternative formulation has the advantage of providing a way to infer the regularity indexes from separate observations of both fragmentation processes, if one was able to measure the energies required to obtain fragments of different close to unit sizes. More precisely,

$$\frac{\log \mathbb{E}(\mathbf{E}(\eta^\lambda)) - \log \mathbb{E}(\mathbf{E}(\eta))}{\log \lambda}$$

should be close to  $\rho$  for  $\eta$  sufficiently close to 1. Alternatively,  $\rho$  could in principle also be deduced from the estimation method of  $\phi$  developed in [9].

In the same vein, we remark that the existence of the limit  $Q_{\phi, \hat{\phi}}$  is equivalent to

$$\exists Q := \lim_{\eta \rightarrow 1^-} \frac{\mathbb{E}(\mathbf{E}(\eta))}{\mathbb{E}(\hat{\mathbf{E}}(\eta))} = \lim_{\eta \rightarrow 1^-} \frac{C}{\widehat{C}} Q_{\phi, \hat{\phi}}.$$

In general, Lemma 4 indeed shows that

$$\widehat{C}(Q^- - 1) \leq \liminf_{\eta \rightarrow 1^-} \frac{\Psi(\ell(\eta)) - \widehat{\Psi}(\ell(\eta))}{\widehat{L}(\ell(\eta))\ell(\eta)^{\widehat{\rho}}} \leq \limsup_{\eta \rightarrow 1^-} \frac{\Psi(\ell(\eta)) - \widehat{\Psi}(\ell(\eta))}{\widehat{L}(\ell(\eta))\ell(\eta)^{\widehat{\rho}}} \leq \widehat{C}(Q^+ - 1),$$

where

$$Q^+ := \limsup_{\eta \rightarrow 1^-} \frac{\mathbb{E}(\mathbf{E}(\eta))}{\mathbb{E}(\hat{\mathbf{E}}(\eta))} = \frac{C}{\widehat{C}} Q_{\phi, \hat{\phi}}^+,$$

and

$$Q^- := \liminf_{\eta \rightarrow 1^-} \frac{\mathbb{E}(\mathbf{E}(\eta))}{\mathbb{E}(\hat{\mathbf{E}}(\eta))} = \frac{C}{\widehat{C}} Q_{\phi, \hat{\phi}}^-.$$

We recall now that, under our assumptions on the Laplace exponents  $\widetilde{\phi}$  and  $\widehat{\widehat{\phi}}$ , by the Dynkin-Lamperti Theorem it weakly holds as  $\eta \rightarrow 1^-$  that

$$\widetilde{\mathbb{P}} \left( \frac{\xi_{T_\eta} - \ell(\eta)}{\ell(\eta)} \in dy \right) \rightarrow \mu(dy) := \frac{\sin(\rho\pi)}{\pi} \frac{dy}{(1+y)y^\rho}$$

and

$$\widetilde{\mathbb{P}} \left( \frac{\widehat{\xi}_{\widehat{T}_\eta} - \ell(\eta)}{\ell(\eta)} \in dy \right) \rightarrow \widehat{\mu}(dy) := \frac{\sin(\widehat{\rho}\pi)}{\pi} \frac{dy}{(1+y)y^{\widehat{\rho}}}.$$

This suggest us the way in which  $\eta$  and  $\eta_0$  should go to 1 in order to observe a coherent close-to-unit size asymptotic behavior. In all the sequel  $\gamma > 1$  is a fixed parameter, and we assume that

$$\eta_0 = \eta_0(\eta) = \eta^\gamma.$$



We have the following

**Lemma 5**

$$\begin{aligned} \lim_{\eta \rightarrow 1^-} \frac{\widetilde{\mathbb{E}} \left( \mathbb{1}_{\xi_{T_\eta} < \ell(\eta_0)} e^{(\alpha - \widehat{\beta})\xi_{T_\eta}} \widehat{\Psi}(\ell(\eta_0) - \xi_{T_\eta}) \right) - \widetilde{\mathbb{E}} \left( \mathbb{1}_{\widehat{\xi}_{\widehat{T}_\eta} < \ell(\eta_0)} e^{(\widehat{\alpha} - \widehat{\beta})\widehat{\xi}_{\widehat{T}_\eta}} \widehat{\Psi}(\ell(\eta_0) - \widehat{\xi}_{\widehat{T}_\eta}) \right)}{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))} \\ = \widehat{C} \left[ \int_0^{\gamma-1} (\gamma - 1 - y)^{\widehat{\rho}} \mu(dy) - \int_0^{\gamma-1} (\gamma - 1 - y)^{\widehat{\rho}} \widehat{\mu}(dy) \right]. \quad (11) \end{aligned}$$

Moreover, in the case  $\frac{1}{2} \geq \rho > \widehat{\rho}$ , the limit is a nonnegative and increasing function of  $\gamma$  for  $\gamma \in [1, 2]$ , which goes to 0 when  $\gamma \rightarrow 1^+$ .

**Proof.** Denote by  $\partial(\eta)$  the numerator in the left hand side (11) and respectively by  $\mu^\eta$  and  $\widehat{\mu}^\eta$  the laws of

$$\frac{\xi_{T_\eta} - \ell(\eta)}{\ell(\eta)} \quad \text{and} \quad \frac{\widehat{\xi}_{\widehat{T}_\eta} - \ell(\eta)}{\ell(\eta)}.$$

We then easily see that

$$\begin{aligned} \partial(\eta) &\leq e^{\gamma \ell(\eta) |\alpha - \widehat{\beta}|} \int_0^{\gamma-1} \widehat{\psi}(\ell(\eta)(\gamma - 1 - y)) \mu^\eta(dy) \\ &\quad - e^{-\gamma \ell(\eta) |\widehat{\alpha} - \widehat{\beta}|} \int_0^{\gamma-1} \widehat{\psi}(\ell(\eta)(\gamma - 1 - y)) \widehat{\mu}^\eta(dy) \end{aligned}$$

and

$$\begin{aligned} \partial(\eta) &\geq e^{-\gamma \ell(\eta) |\alpha - \widehat{\beta}|} \int_0^{\gamma-1} \widehat{\psi}(\ell(\eta)(\gamma - 1 - y)) \mu^\eta(dy) \\ &\quad - e^{\gamma \ell(\eta) |\widehat{\alpha} - \widehat{\beta}|} \int_0^{\gamma-1} \widehat{\psi}(\ell(\eta)(\gamma - 1 - y)) \widehat{\mu}^\eta(dy). \end{aligned}$$

On the other hand, by similar estimates as in the previous lemma, one checks that

$$\theta(x) := \frac{\widehat{\psi}(x)}{\widehat{C} \widehat{L}(x) x^{\widehat{\rho}}} \rightarrow 1 \quad (12)$$

when  $x \searrow 0$ , and thus  $\theta(x)$  is slowly varying at 0. Fix now  $\varepsilon \in (0, 1)$ , and recall that for a slowly varying at 0 function  $G(x)$ , the convergence  $G(\lambda x)/G(x) \rightarrow 1$  is uniform in  $\lambda \in [0, \lambda_0]$  for all  $\lambda_0 \in (0, 1)$ . Therefore, since

$$\widehat{\psi}(\ell(\eta)y) = \frac{\theta(\ell(\eta)y)}{\theta(\ell(\eta))} \frac{\widehat{L}(\ell(\eta)y)}{\widehat{L}(\ell(\eta))} \widehat{\psi}(\ell(\eta))y^{\widehat{\rho}},$$

we deduce that if  $\eta \in (0, 1)$  is sufficiently close to 1,

$$\forall y \in [0, \gamma - 1] : \quad (1 - \varepsilon)\widehat{\psi}(\ell(\eta))y^{\widehat{\rho}} \leq \widehat{\psi}(\ell(\eta)y) \leq (1 + \varepsilon)\widehat{\psi}(\ell(\eta))y^{\widehat{\rho}}.$$

Moreover, from (12), it follows that if  $\eta$  is sufficiently close to 1 then

$$\forall y \in [0, \gamma - 1] : \quad \widehat{C}(1 - \varepsilon)^2 y^{\widehat{\rho}} \leq \frac{\widehat{\psi}(\ell(\eta)y)}{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))} \leq \widehat{C}(1 + \varepsilon)^2 y^{\widehat{\rho}}. \quad (13)$$

It follows that

$$\limsup_{\eta \rightarrow 1^-} \frac{\partial(\eta)}{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))} \leq (1 + \varepsilon)^2 \widehat{C} A_\gamma - (1 - \varepsilon)^2 \widehat{C} \widehat{A}_\gamma$$

and

$$\liminf_{\eta \rightarrow 1^-} \frac{\partial(\eta)}{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))} \geq (1 - \varepsilon)^2 \widehat{C} A_\gamma - (1 + \varepsilon)^2 \widehat{C} \widehat{A}_\gamma,$$

where

$$A_\gamma = \frac{\sin(\pi\rho)}{\pi} \int_0^{\gamma-1} (\gamma-1-u)^{\widehat{\rho}} \frac{du}{(1+u)u^\rho}, \quad \widehat{A}_\gamma = \frac{\sin(\pi\widehat{\rho})}{\pi} \int_0^{\gamma-1} (\gamma-1-u)^{\widehat{\rho}} \frac{du}{(1+u)u^{\widehat{\rho}}}.$$

The first statement follows by letting  $\varepsilon \rightarrow 0^+$ . The asserted properties of  $\widehat{C}(A_\gamma - \widehat{A}_\gamma)$  are consequence of the inequalities  $u^{-\rho} > u^{-\widehat{\rho}}$  for  $u \in (0, 1)$ ,  $\sin(\pi\rho) > \sin(\pi\widehat{\rho}) > 0$  when  $\frac{1}{2} > \rho > \widehat{\rho}$ , and dominated convergence. ■

We next introduce helpful concepts in order to state our results on the energy for large thresholds.

**Definition 1** (i) *The fragmentation processes  $\mathbf{X}$  is said to be infinitesimally efficient (inf. eff.) compared to  $\widehat{\mathbf{X}}$  if  $(\mathbf{RV})$  holds and  $Q_{\phi, \widehat{\phi}}^+ < \frac{\widehat{C}}{C}$ .*

*Conversely,*

(ii) *The fragmentation processes  $\widehat{\mathbf{X}}$  is said to be inf. eff. compared to  $\mathbf{X}$  if  $(\mathbf{RV})$  holds and  $Q_{\phi, \widehat{\phi}}^- > \frac{\widehat{C}}{C}$ .*

For instance,  $\mathbf{X}$  is inf. eff. compared to  $\widehat{\mathbf{X}}$  if  $\rho > \widehat{\rho}$  or if  $\rho = \widehat{\rho}$  and  $Q_{\phi, \widehat{\phi}}$  exists in  $[0, \frac{\widehat{C}}{C})$ . Similarly,  $\widehat{\mathbf{X}}$  is inf. eff. compared to  $\mathbf{X}$  e.g. if  $\rho < \widehat{\rho}$  or if  $\rho = \widehat{\rho}$  and  $Q_{\phi, \widehat{\phi}}$  exists in  $(\frac{\widehat{C}}{C}, \infty]$ .

**Remark 6** *We observe that  $\mathbf{X}$  (respectively  $\widehat{\mathbf{X}}$ ) is inf. eff. compared to  $\widehat{\mathbf{X}}$  (respectively  $\mathbf{X}$ ) if and only if  $\mathbb{E}(\mathbf{E}(e^{-x}))$  and  $\mathbb{E}(\widehat{\mathbf{E}}(e^{-x}))$  are regularly varying functions at  $x = 0$  with indexes in  $(0, 1)$  and  $Q^+ = \limsup_{\eta \rightarrow 1^-} \frac{\mathbb{E}(\mathbf{E}(\eta))}{\mathbb{E}(\widehat{\mathbf{E}}(\eta))} < 1$  (respectively  $Q^- = \liminf_{\eta \rightarrow 1^-} \frac{\mathbb{E}(\mathbf{E}(\eta))}{\mathbb{E}(\widehat{\mathbf{E}}(\eta))} > 1$ ).*

Bringing all together, we have obtain:

**Theorem 4** *(Two-step procedure versus second fragmentation only)*

*For each  $\gamma \in (1, \infty)$  it holds:*

(a) *If  $\widehat{\mathbf{X}}$  is inf. eff. compared to  $\mathbf{X}$  and  $Q^- = Q = \infty$  (in particular if  $\widehat{\rho} > \rho$ ), then:*

$\forall M > 0 \ \exists \eta_M \in (0, 1)$  *such that*

$$\forall \eta \in (\eta_M, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) > \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + M\mathbb{E}(\widehat{\mathbf{E}}(\eta)) > \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)).$$

(b) *If  $\widehat{\mathbf{X}}$  is inf. eff. compared to  $\mathbf{X}$  and  $Q^- \in (1, \infty)$  (and thus  $\rho = \widehat{\rho}$ ), then:*

$\forall \varepsilon \in (0, Q^- - 1) \exists \eta_\varepsilon \in (0, 1)$  such that

$$\forall \eta \in (\eta_\varepsilon, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) > \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (Q^- - 1 - \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)) > \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)).$$

(c) If  $\mathbf{X}$  is inf. eff. compared to  $\widehat{\mathbf{X}}$  and  $Q^+ \in (0, 1)$  (and thus  $\rho = \widehat{\rho}$ ), then:

$\forall \varepsilon \in (0, 1 - Q^+) \exists \eta_\varepsilon \in (0, 1)$  such that

$$\forall \eta \in (\eta_\varepsilon, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (Q^+ - 1 + \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)).$$

(d) If  $\mathbf{X}$  is inf. eff. compared to  $\widehat{\mathbf{X}}$  and  $Q^+ = Q = 0$  (in particular if  $\widehat{\rho} < \rho$ ), then:

$\forall \varepsilon \in (0, 1), \exists \eta_\varepsilon \in (0, 1)$  such  $\forall \eta \in (\eta_\varepsilon, 1]$ :

$$\mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (A_\gamma - \widehat{A}_\gamma - 1 - \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)) < \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (A_\gamma - \widehat{A}_\gamma - 1 + \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)).$$

(The quantities  $A_\gamma$  and  $\widehat{A}_\gamma$  were defined in Lemma 5).

Moreover, if  $\frac{1}{2} \geq \rho > \widehat{\rho}$ ,  $\exists \gamma_0 \in (1, 2]$  such that  $\forall \gamma \in (1, \gamma_0]$ , one has  $1 - A_\gamma + \widehat{A}_\gamma > 0$  and

$\forall \varepsilon \in (0, 1 - A_\gamma + \widehat{A}_\gamma)$ ,

$$\forall \eta \in (\eta_\varepsilon, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (A_\gamma - \widehat{A}_\gamma - 1 + \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)).$$

In all four cases, similar statements hold with  $\mathbb{E}(\widehat{\mathbf{E}}(\eta))$  replaced by  $\widehat{C} \left[ \widehat{\phi} \left( \frac{1}{\log(1/\eta)} \right) \right]^{-1}$ .

**Proof.** By Remark 2 and the previous results, we simply have to notice that when  $\eta \rightarrow 1^-$ ,

$$\frac{\mathbb{E}(\widehat{\mathbf{E}}(\eta))}{\widehat{C}} \sim \widehat{L}(\ell(\eta))(\ell(\eta))^{\widehat{\rho}} = \left[ \widehat{\phi} \left( \frac{1}{\ell(\eta)} \right) \right]^{-1},$$

$\mathcal{E}(1, \eta^\gamma) = \widehat{\mathbf{E}}(\eta^\gamma)$  and the quantities  $A_\gamma$  and  $\widehat{A}_\gamma$  are equal if  $\rho = \widehat{\rho}$ . The last assertion in part

(d) is consequence of the last part of Lemma 5.

■

The previous theorem provided conditions on large thresholds  $\eta$  and  $\eta_0$  under which the use of the second fragmentation process can be told to be efficient or not. We next briefly address the efficiency of using or not the first fragmentation process. The arguments of the following theorem are similar to those of the previous lemmas, so we just sketch its proof.

We use the following notation

$$\forall \gamma \in (1, \infty) : \quad B_\gamma := \frac{\sin(\pi\rho)}{\pi} \int_0^{\gamma-1} (\gamma-1-u)^\rho \frac{du}{(1+u)u^\rho}.$$

**Theorem 5** (*Two-step procedure versus first fragmentation only*)

For all  $\gamma \in (1, \infty)$  it holds:

(a) If  $\widehat{\mathbf{X}}$  is inf. eff. compared to  $\mathbf{X}$ , then:

$\forall \varepsilon \in (0, 1 - \frac{1}{Q^-}) \quad \exists \eta_\varepsilon \in (0, 1)$  such that

$$\forall \eta \in (\eta_\varepsilon, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\mathbb{E}(\eta^\gamma)) + \left( \frac{1}{Q^-} - 1 + \varepsilon \right) B_\gamma \mathbb{E}(\mathbb{E}(\eta)) < \mathbb{E}(\mathbb{E}(\eta^\gamma)).$$

(b) If  $\mathbf{X}$  is inf. eff. compared to  $\widehat{\mathbf{X}}$  and  $Q^+ \in (0, 1)$  (and thus  $\rho = \widehat{\rho}$ ), then:

$\forall \varepsilon \in (0, \frac{1}{Q^+} - 1)$ ,  $\exists \eta_\varepsilon \in (0, 1)$  such that

$$\forall \eta \in (\eta_\varepsilon, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) > \mathbb{E}(\mathbb{E}(\eta^\gamma)) + \left( \frac{1}{Q^+} - 1 - \varepsilon \right) B_\gamma \mathbb{E}(\mathbb{E}(\eta)) > \mathbb{E}(\mathbb{E}(\eta^\gamma)).$$

(c) If  $\mathbf{X}$  is inf. eff. compared to  $\widehat{\mathbf{X}}$  and  $Q^+ = Q = 0$  (in particular if  $\widehat{\rho} < \rho$ ), then:

$\forall M > 0 \quad \exists \eta_M \in (0, 1)$  such that

$$\forall \eta \in (\eta_M, 1] : \quad \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) > \mathbb{E}(\mathbb{E}(\eta^\gamma)) + M \mathbb{E}(\mathbb{E}(\eta)) > \mathbb{E}(\mathbb{E}(\eta^\gamma)).$$

In all cases, one can replace  $\mathbb{E}(\mathbf{E}(\eta))$  by  $C \left[ \tilde{\phi} \left( \frac{1}{\ell(\eta)} \right) \right]^{-1}$ .

**Proof.** Fix  $\gamma > 1$  and  $\varepsilon \in (0, 1)$ . As in Lemma 5 we get that for all  $y \in [0, \gamma - 1]$ ,

$$C(1 - \varepsilon)^2 y^\rho \leq \frac{\psi(\ell(\eta)y)}{(\ell(\eta))^\rho L(\ell(\eta))} \leq C(1 + \varepsilon)^2 y^\rho$$

and

$$\widehat{C}(1 - \varepsilon)^2 \frac{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))}{(\ell(\eta))^\rho L(\ell(\eta))} y^{\widehat{\rho}} \leq \frac{\widehat{\psi}(\ell(\eta)y)}{(\ell(\eta))^\rho L(\ell(\eta))} \leq \widehat{C}(1 + \varepsilon)^2 \frac{(\ell(\eta))^{\widehat{\rho}} \widehat{L}(\ell(\eta))}{(\ell(\eta))^\rho L(\ell(\eta))} y^{\widehat{\rho}}$$

if  $\eta$  is close enough to 1. Set now  $\overline{\partial}(\eta) := \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma) - \mathcal{E}(\eta^\gamma, \eta^\gamma))$ . From the previous bounds, and from the explicit expression for  $\overline{\partial}(\eta)$  given in Remark 2, we deduce that

$$C \left( \frac{A_\gamma}{Q^+} - B_\gamma \right) \leq \liminf_{\eta \rightarrow 1^-} \frac{\overline{\partial}(\eta)}{(\ell(\eta))^\rho L(\ell(\eta))} \leq \limsup_{\eta \rightarrow 1^-} \frac{\overline{\partial}(\eta)}{(\ell(\eta))^\rho L(\ell(\eta))} \leq C \left( \frac{A_\gamma}{Q^-} - B_\gamma \right).$$

Part (a) follows from this relation, using the facts that  $A_\gamma = B_\gamma$  if  $\rho = \widehat{\rho}$ , and that  $Q_- = \infty$  if  $\widehat{\rho} > \rho$ . The remaining parts are similar. ■

**Remark 7** If  $\rho = \widehat{\rho}$  and  $Q \in (0, \infty)$  exists, one obtains for  $\eta$  close enough to 1,

$$\mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (Q - 1 - \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta)) < \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) + (Q - 1 + \varepsilon)\mathbb{E}(\widehat{\mathbf{E}}(\eta))$$

$$\mathbb{E}(\mathbf{E}(\eta^\gamma)) + (Q^{-1} - 1 - \varepsilon)B_\gamma \mathbb{E}(\mathbf{E}(\eta)) < \mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) < \mathbb{E}(\mathbf{E}(\eta^\gamma)) + (Q^{-1} - 1 + \varepsilon)B_\gamma \mathbb{E}(\mathbf{E}(\eta)).$$

In particular, when  $Q = 1$  we deduce that  $\mathbb{E}(\mathcal{E}(\eta, \eta^\gamma)) \sim \mathbb{E}(\widehat{\mathbf{E}}(\eta^\gamma)) \sim \mathbb{E}(\mathbf{E}(\eta^\gamma))$  when  $\eta \rightarrow 1^-$ , as one could expect.

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